INDEPENDENCE RESULTS IN CONCRETE MATHEMATICS

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A project proposal to undertake research in Foundations of Mathematics as a postdoctoral researcher in Bristol University.

This project proposal describes my research plans for the near future. It is devoted to the programme in Foundations of Mathematics which dates back to K. Gödel's discovery of Incompleteness. The main goals can be grouped as the eleven concrete tasks below. Ramsey-theoretic, well-order-theoretic and well-quasi-order-theoretic considerations have always been the usual sources of unprovability results, since the pioneering discoveries of J. Paris and H. Friedman. We are going to obtain independence results in these and other parts of 'concrete' mathematics: Ramsey theory, well-quasi-order theory, analytic combinatorics, braid theory, number theory, geometry of manifolds, ergodic theory and the theory of chaotic dynamical systems.

In each of the eleven tasks, there are already some preliminary results and it is somehow clear how to obtain final theorems.

Objective	Preliminary results and how to proceed
Task 1: A new Riemann zeta-function conjecture (joint with A. Weiermann)	
Find a new series of unprovable statements and convincing conjectures about the Riemann zeta-function, e.g., using the famous correspondence between zeros of the zeta function and random matrix ensembles.	A series of unprovability results about the Riemann zeta-function has been proved in [8], using almost periodicity, probabilistic arguments in the critical strip and Voronin's universality theory. This is a very fruitful direction with impressive independence results, appealing to a large audience.
Task 2: Manifolds and ordered fundamental groups	
Adapt and extend $I\Sigma_2$-unprovability results about braids to other geometric subjects where braids play an important role. Find unprovable statements about fundamental groups of 3-manifolds, transfer the treatment of braids to other orderable groups. A further attempt will be made to find unprovable statements with some amount of 'physical' meaning (by studying the use of braids in physics).	The origin of this theme goes back to Dehornoy's left-invariant ordering of the positive part of the Artin's braid group as ω^{ω^ω}, [14]. There have been several recent results by L. Carlucci and the author in [7] on $I\Sigma_2$-unprovable statements about braid groups. Many of these theorems seem to have analogues for other left-orderable groups, including some fundamental groups of 3-dimensional manifolds. This Task was inspired by [14] and [13].

Objective	Preliminary results and how to proceed
Task 3: Indiscernibles	
Explore further possibilities with indiscernibles to produce new unprovability results of ramseyan nature and, hopefully, of number-theoretic nature.	The author has a lot of experience proving unprovability using indiscernibles (see [1], [4], [11]). There is much more to say about this method, beyond the Paris-Harrington Principle (including a story with number-theoretic flavour).
Task 4: Building models of strong theories	
Build models of strong theories (e.g. ZF or extensions of ZF) directly 'by hands' to obtain unprovable assertions. (I will be interested in both first-order and second-order arithmetical unprovable statements.)	Clearly, this will be a construction of a countable directed graph. During the construction, we shall be forced to employ some combinatorially needed assumptions to make our construction work. These will be the unprovable statements we are seeking. First examples were given by H. Friedman (see e.g. [16]).
Task 5: Independence results about games	
Build models by games in order to obtain new independence results (e.g. that certain games don't provably have a winning strategy). We are planning to formulate a generalised game of Noughts and Crosses on a many-dimensional board such that a winning diagonal would be translatable into a desired set of indiscernibles. (It may turn out to be a version of the Hales-Jewett Theorem.) Many other games can be invented with a similar property.	The method of indicators of J. Paris often works as a game between two players where Player I tries to ensure that the final initial segment between two points of a model is a model of PA and Player II tries to prevent it. The game of finite (nonstandard) length is determined, so it turns out that if a set is large enough to accommodate a strong cut then Player I has a winning strategy, otherwise Player II wins. It is only necessary to translate such a game into children's language.
Task 6: Dynamical Systems and Ergodic Theory (joint with A. Weiermann)	
Ergodic theory and dynamical systems are the two subjects where strong and unprovable statements are most easily cropping up in existing set-up. Find further unprovable statements in these areas. Exploit the deep connections of these areas with the theory of uniform distribution of sequences modulo 1 and simultaneous diophantine approximation.	An unprovable statement about the logistic mapping with large parameter (a chaotic dynamical system) can be found in [8] as well as some discussion of how to use the mathematically deep phenomenon connecting chaos, equidistribution and simultaneous [diophantine] approximation to obtain independence results.
Task 7: Universality (joint with A. Weiermann)	
It was noticed (in [3] and in [8] in the context of Voronin-universality) that almost any situation where all possible patterns are already present in one complex object (e.g. when every function of a certain class is approximated by some universal function on a certain subset) leads to unprovability results by clever encoding of Ramsey-style style unprovable statements.	Here, we aim for find more existing universality phenomena in several branches of mathematics (e.g. in complex analysis and in p-adic analysis) and convert them into unprovability results. The author has a few preliminary results (including a p-adic result). It is important to try to reformulate the statements in a Ramsey-free way, for more public appeal.

Objective	Preliminary results and how to proceed
Task 8: My own Π^0_1 statement unprovable in a strong theory	
Although concentrating on finding especially Π^0_1-unprovable statements may be difficult, and these statements may be less appealing, I am planning to invent my own examples and my own methods of obtaining them.	Examples of unprovable Π^0_1 statements follow from Gödel's theorems, from the work of H. Friedman, from MDRP-theorem [18] (and possibly from some S. Shelah's work). This is a difficult problem if we want to have beautiful combinatorial examples. One way of thinking could be to say "all finite sequences of natural numbers of a certain shape approximate a model of arithmetic to a certain degree (which depends on the sequence)".
Task 9: Model theory of trees	
In a model M of arithmetic, continue the hierarchy of cuts introduced by J. Paris to include new kinds of cuts corresponding to ACA_0^+, ATR_0 and other theories, for example 'Kruskal cuts' (every M-coded I-unbounded set of M-finite trees has an unbounded increasing subsequence) for different numbers of labels and branching of trees. Develop full indicator theory in this context and beyond.	This theme can be thought of as D. Schmidt's book [20] crossed with Paris-Kirby indicators. Apart from easily-expected results (does every nonstandard model of $I\Sigma_1$ have continuum-many Kruskal cuts?) we expect to end up with a fully-functioning technology to produce logical strength results at the very high pitches of consistency strength. Some early parts of this project may look like a translation of the story of maximal well-ordered linearisations but the eventual model-theoretic understanding of Kruskal's theorem (and beyond) will be very rewarding. I am expecting many new independence results to spring here.
Task 10: Infinitary Ramsey Theory and Functional Analysis	
Establish logical strength of several infinitary statements from modern Ramsey Theory (see [12]): e.g. award-winning theorems by T. Gowers, some statements about blocks and barriers, about strategically Ramsey sets and about oscillation stability. Geometric Functional analysis also provides us with many excellent candidates: a version of Dvoretzki's theorem [19] may be strong.	In [5] and [6], the author gave model-theoretic proofs of logical strength of several second-order Ramsey-like statements. The three ways to establish logical strength of new infinitary ramseyan principles of [12] are: mutual implications with principles of known or unknown strength (e.g., RT, RT^1_n, RT^3, RT^2_2, etc) model-theoretic constructions (as in [5] or [17]), and the density approach as in [5]. This Task amounts to developing the model-theoretic side of Reverse Mathematics.
Task 11: Hypothesis H about prime numbers	
This very general and extremely strong statement is begging for an unprovability proof: for any finite number of polynomials $p_1(x), \ldots, p_n(x)$ without a prime number dividing their product, they are simultaneously prime on an infinite set. Even its weaker predecessor, Buniakovskiy's Conjecture (1857) "every irreducible polynomial assumes infinitely-many prime values" may turn out to be unprovable.	It is very clear what should be the first few steps in the unprovability proof (these will include converting Δ_0-formulas into polynomials, encoding generic colourings using 'randomness' results about primes and indiscernibility arguments). This may be a difficult problem. Even if the problem does not get solved, the by-products should still be very exciting, e.g. the case of polynomials of several variables seems much easier but still would be very impressive. There are some preliminary results in [11].

Here are some smaller questions, whose solution might be easier than that of the Tasks above:

(1) a study of Diophantine games from [18] (and corresponding unprovable statements, as in [11]);
(2) an α-large approach to Kruskal's theorem (given an ordinal β, what is the minimal α such that every α-large sequence of trees has an increasing β-large increasing subsequence?);
(3) a fresh look at the famous analogy between arithmetical unprovable statements and large cardinals [4] (try both sides of this analogy: model-theoretic and J. Ketonen-style);
(4) extension of the author's results about exact unprovability results for compound combinatorial classes in [9] to the yet unresolved mysterious cases where the "supercriticality" condition fails and to recursive and implicit specifications;
(5) the study (joint with A. Weiermann) of densities for the Erdös-Moser principle [6] (and of the interesting phenomenon that occurs here when two weak statements, adjoined together become strong);
(6) the study of unprovability of well-quasi-orderedness of supertrees, hierarchies, alcohols, mobiles, unary-binary trees and $\leq k$-branching plane trees, finding the exact versions of these results in the spirit of [21];
(7) mending the gap between upper and lower bounds in the author's result about graph minors in [15], the study of open questions about graph minors in [15].

The study of Unprovability is an undertaking of enormous general intellectual importance, relevant not only throughout mathematics but to general questions in philosophy and methodology of science and 'rigorous thinking'.

Apart from the particularly difficult tasks 4, 8 and 11, I am planning to be able to complete the whole project within 30-36 months. The difficulty of tasks 4, 8 and 11 can't be predicted at the moment but a full-scale attempt to solve these problems will be made.

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